

DEVIATION FROM THE EINSTEIN RELATION OF THE SINGLE-FILE DIFFUSION COEFFICIENT

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The deviation of the diffusion coefficient D of a one-dimensional fluid from its tracer diffusion coefficient D^* is calculated from Gürsey's equation of state, using a square-well interaction potential. Enormous departures from the Einstein relation $D = D^*$ are noted.

1. Introduction

In a classic paper Hodgkin and Keynes [1] demonstrated the existence of strong interactions among potassium ions as they traverse the membrane of the giant squid axon. Their method was to measure, by means of radioactive tracers, the influx and efflux of potassium across a membrane poisoned to eliminate active transport, and to observe that the measured influx–efflux ratio did not satisfy a relation for passive transport generally valid in the absence of interactions among diffusing molecules. For electrically uncharged molecules, the relation reduces to the statement that the influx–efflux ratio equals the ratio of concentrations on the two sides of the membrane, while for an ionic species it is necessary to account as well for the potential difference across the membrane. The criterion for validity of the relation is negligible interaction among molecules of the diffusing species within the membrane, a condition satisfied if its concentration in the membrane is low. Hodgkin and Keynes, therefore, interpreted the massive deviation of their data from this “independence” relation as suggesting that potassium ions are concentrated in channels so narrow that the movement of a given ion is blocked by the ions on either side of it – the ions cannot pass each other and must move in single file.

Because a basic theoretical framework for single-file kinetics appears to be lacking, it is of interest to consider the behavior of the diffusion coefficient of a one-dimensional fluid, for which the molecular mo-

tion must of necessity be “single-file”. Based on Gürsey's exact equation of state [2], the expression derived here gives a representation of the deviation of the diffusion coefficient from the Einstein relation for non-interacting particles. The treatment in this article is restricted to electrically neutral molecules, interacting through short-range forces, and is worked out explicitly for a square-well binary interaction potential. Hopefully, at least an approximate calculation for ionic particles can be presented at a later time.

Before proceeding, a qualitative summary of the results obtained may be useful:

(1) Fick's first law holds with a diffusion coefficient dependent on the parameters of the interaction potential and the linear concentration (number of molecules per unit length).

(2) Depending on the parameters of the interaction potential, the deviation from the Einstein “independence” relation may be either positive, indicating a net repulsive interaction, or negative, indicating a net attractive interaction. Therefore, over and above its random thermal motion, a given molecule may be directed by a net intermolecular force either toward or away from the side of greater linear concentration.

(3) Even for linear concentrations low relative to closest packing, deviations from the Einstein relation may be enormous.

(4) The rate of tracer diffusion is extremely sensitive to the presence of a macroscopic gradient, and even mild conditions can result in tracer flow in opposition to the tracer gradient (strong coupling between

tracer and non-tracer flows).

2. Theory

2.1. Diffusion in a one-dimensional fluid

The general analysis of diffusion reviewed in this section is well known and applicable to any dimension, although the discussion is in terms of one dimension. Suppose the molecules of a one-component fluid are distributed in one dimension, along the x -axis. Their flux is given by

$$j = -L(d\lambda/dx), \quad (1)$$

where λ is the chemical potential, or Gibbs free energy per molecule (it is convenient to reserve the more conventional symbol μ for a three-dimensional system [3]), and L is a coefficient independent of the gradient of chemical potential. Because the fluid is one-dimensional, j has units molecule sec^{-1} , the number of molecules crossing a point in unit time. The chemical potential is given by an expression of the form,

$$\lambda = \bar{\lambda}(T) + kT \ln \rho + \lambda^{\text{ex}}(T, \rho) + \psi, \quad (2)$$

where $\bar{\lambda}$ is determined by the properties of an isolated molecule, ρ is the linear concentration with units molecule cm^{-1} , λ^{ex} is the "excess" chemical potential due to the mean potential energy of interaction among the molecules, and ψ is the potential energy per molecule generated by any externally applied force. By convention, the excess chemical potential may be written as

$$\lambda^{\text{ex}} = kT \ln \gamma, \quad (3)$$

and γ is called the activity coefficient. Because λ^{ex} , or γ , accounts for molecular interactions, which become negligible as the concentration becomes small,

$$\lim_{\rho \rightarrow 0} \lambda^{\text{ex}} = 0, \quad \lim_{\rho \rightarrow 0} \gamma = 1. \quad (4)$$

For an isothermal system, then,

$$j = -L \left(\frac{kT}{\rho} + \frac{d\lambda^{\text{ex}}}{d\rho} \right) \frac{d\rho}{dx} - L \frac{d\psi}{dx}. \quad (5)$$

If a molecule has an average directed velocity v , the corresponding contribution to the flux is ρv . If f is the force responsible for the directed velocity, then $f = \zeta v$, where ζ is the friction coefficient. Since $-(d\psi/dx)$ is

such a force, it follows that L in eq. (5) equals ρ/ζ . Hence,

$$j = -\frac{kT}{\zeta} \left(1 + \rho \frac{d\lambda^{\text{ex}}/kT}{d\rho} \right) \frac{d\rho}{dx} - \frac{\rho}{\zeta} \frac{d\psi}{dx}. \quad (6)$$

The influence of externally applied forces will not be considered further in this paper, so $d\psi/dx$ may be set equal to zero, and, if eq. (3) is used,

$$j = -(kT/\zeta)(1 + d \ln \gamma / d \ln \rho) (d\rho/dx). \quad (7)$$

Fick's first law is,

$$j = -D(d\rho/dx), \quad (8)$$

where D is called the diffusion coefficient. Comparison with eq. (7) yields,

$$D = (kT/\zeta)(1 + d \ln \gamma / d \ln \rho). \quad (9)$$

Now consider the flux j^* of a tracer species with the specification that the macroscopic concentration is uniform along the pore. That is, $d\rho^*/dx \neq 0$ but $d\rho/dx = 0$, where ρ^* is the tracer concentration. For the chemical potential of the tracer species, one has

$$\lambda^* = \bar{\lambda}(T) + kT \ln \rho^* + \lambda^{\text{ex}}(T, \rho) + \psi, \quad (10)$$

where $\bar{\lambda}$, λ^{ex} , and ψ are the same functions that appear in eq. (2). In particular, λ^{ex} , the free energy per tracer molecule due to intermolecular interaction, is a function of ρ , not of ρ^* , since a tracer molecule interacts with *all* the molecules in its vicinity. Then, since $d\lambda^{\text{ex}}/dx = 0$,

$$j^* = -L^* (d\lambda^*/dx) = -\frac{kT}{\rho^*} L^* \frac{d\rho^*}{dx} - L^* \frac{d\psi}{dx}, \quad (11)$$

The second term on the right-hand side must equal $-(\rho^*/\zeta)(d\psi/dx)$, so that L^* is identified as ρ^*/ζ ; and, if $d\psi/dx$ is set equal to zero,

$$j^* = -(kT/\zeta) (d\rho^*/dx). \quad (12)$$

The tracer diffusion coefficient D^* is defined by Fick's first law as the coefficient of $d\rho^*/dx$ in eq. (12), so

$$D^* = kT/\zeta. \quad (13)$$

From eqs. (9) and (13) the macroscopic diffusion coefficient D is seen to satisfy the relation

$$D/D^* = 1 + d \ln \gamma / d \ln \rho. \quad (14)$$

Applications of eq. (4) to eq. (14) results in the Einstein relation,

$$\lim_{\rho \rightarrow 0} (D/D^*) = 1 \quad (15)$$

The deviation from the Einstein relation at finite concentration is designated by δ ,

$$D/D^* = 1 + \delta. \quad (16)$$

According to eq. (14), then,

$$\delta = d \ln \gamma / d \ln \rho. \quad (17)$$

It is important to understand the physical meaning of δ . In a concentration gradient the environment of a given molecule is asymmetric, so that there also exists a gradient in the average potential energy of interaction with the molecules which surround it. This gradient is given precisely by $d\lambda^{ex}/dx$, or $kT d \ln \gamma / dx$, since λ^{ex} is that part of the chemical potential due to the potential energy of molecular interaction. The negative of the gradient is a net force on the molecule, which causes an average directed velocity to be superposed on its random thermal motion. The contribution of this velocity to the flux is therefore equal to

$$\begin{aligned} -(\rho/\zeta)(d\lambda^{ex}/dx) &= -(kT/\zeta)(d \ln \gamma / d \ln \rho)(d\rho/dx) \\ &= -\delta(kT/\zeta)(d\rho/dx). \end{aligned}$$

For a given concentration gradient, therefore, the magnitude of δ is a measure of the magnitude of the velocity directed by unbalanced intermolecular forces. Its sign is determined by whether the intermolecular forces are, on the average, attractive or repulsive. If attractive, the velocity is directed toward the region of higher concentration, and δ is negative; if repulsive, a molecule is driven away from the side of greater concentration, and δ is positive. Repulsive forces enhance diffusion along the concentration gradient; attractive forces retard it.

The only general restriction on the numerical value of δ is

$$-1 < \delta \leq \infty, \quad (18)$$

in other words, with reference to eq. (9), D must be positive. This restriction arises from the general thermodynamic stability criterion $d\lambda/d\rho > 0$, a condition, which, with eq. (2) for λ , immediately gives (18). Note that an infinite value of δ , or D , is not forbidden on thermodynamic grounds.

2.2. Statistical thermodynamics of a one-dimensional fluid

In order to obtain an explicit expression for δ in terms of ρ and the parameters which characterize the intermolecular potential energy, one must first calculate the partition function for the system and, from that, the free energy and chemical potential. This task was done some time ago by Grsey [2]. His expressions for the partition function, thermodynamic functions, and equation of state are exact subject to the following conditions:

(1) The system is a one-dimensional fluid at equilibrium in the absence of external fields. It consists of N identical molecules distributed along a line of length L , and the results are exact in the usual macroscopic limit for which N and L tend to infinity while the uniform linear concentration $\rho = N/L$ remains constant.

(2) The potential energy of interaction $u(x)$ between two molecules is of finite range x_0 : for $x > x_0$, $u(x) = 0$.

(3) The molecules are impenetrable: for some value a , the "diameter" of a molecule, $u(x) = \infty$ if $x < a$.

(4) The intermolecular force operating between a pair of molecules is of such short range that all interactions except those between nearest neighbors can be neglected. This condition is assured if $x_0 < 2a$.

To evaluate the partition function, Grsey used the convolution theorem for the Laplace transform, a technique that works precisely because of the "single-file" nature of the system and that therefore cannot be applied to systems of higher dimensionality. The expression for the partition function Q is

$$Q = (2\pi mkT/h^2)^{N/2} \exp(\pi L/kT) [\phi(T, \pi)]^N, \quad (19)$$

where m is the mass of a molecule, h is Planck's constant,

$$\phi(T, \pi) = \int_0^\infty \exp\{-[\pi x + u(x)]/kT\} dx, \quad (20)$$

and π is the "pressure" defined in analogy to the three-dimensional case by the appropriate partial derivative of the Helmholtz free energy F^* . The latter is related in the usual way to the partition function,

* In the first factor of the expression for Q , $\pi = 3.1416 \dots$. This double use of the symbol π will not cause confusion.

$$F = -kT \ln Q \quad (21)$$

and

$$\pi = -(\partial F / \partial L)_{T, N} \quad (22)$$

Note from its units that π is really a force, not a pressure in the three-dimensional sense of force per unit area. The Gibbs free energy is $G = F + \pi L$ and the chemical potential is $\lambda = G/N$. Hence,

$$\lambda = -\frac{1}{2} kT \ln(2\pi m kT/h^2) - kT \ln \phi(T, \pi) \quad (23)$$

The first term on the right-hand side of eq. (23) is recognized as a contribution from the single translational degree of freedom available to a molecule; it is therefore to be identified with $\bar{\lambda}(T)$ in eq. (2).

$$\lambda = \bar{\lambda}(T) - kT \ln \phi(T, \pi) \quad (24)$$

It will be recalled from eq. (17) that λ is required as an explicit function of ρ rather than of π . Therefore the equation of state $\pi = \pi(T, \rho)$ is needed. Gürsey showed that the equation of state is contained in the relation

$$1 + \rho kT [\partial \ln \phi(T, \pi) / \partial \pi] = 0 \quad (25)$$

which follows immediately from the thermodynamic identity $L = (\partial G / \partial \pi)_{T, N}$ with $G = N\lambda$ and λ given by eq. (24). At this point the three special cases treated by Gürsey will be reviewed: the case of vanishing potential energy of interaction (ideal fluid), the hard-sphere potential, and the square-well potential.

If $u(x) \equiv 0$, eq. (20) gives the simple result,

$$\phi(T, \pi) = kT/\pi \quad (26)$$

Hence, from eq. (20)

$$\pi = \rho kT \quad (27)$$

which of course, is the equation of state of an ideal gas in one dimension. From eqs. (24), (26), and (27),

$$\lambda = \bar{\lambda}(T) + kT \ln \rho \quad (28)$$

the familiar expression for the chemical potential of an ideal gas. Comparison of eqs. (2) and (28) shows that

$$\lambda^{\text{ex}} = kT \ln \gamma = 0 \quad (29)$$

A hard-sphere fluid is characterized by an interaction potential of the form,

$$u(x) = \infty, \quad x < a \quad (30a)$$

$$u(x) = 0, \quad x > a \quad (30b)$$

Each molecule is a rigid sphere of diameter a ; the intermolecular force is a perfectly elastic repulsion. Eq. (20) yields,

$$\phi(T, \pi) = (kT/\pi) \exp(-\pi a/kT) \quad (31)$$

from which, along with eq. (25), it follows that

$$\pi = \rho kT (1 - a\rho)^{-1} \quad (32)$$

Gürsey's result for the equation of state of a one-dimensional hard-sphere fluid. When $a = 0$, eq. (32) collapses, as it must, to the corresponding relation, eq. (27), for an ideal fluid. From eqs. (24), (31), and (32),

$$\lambda = \bar{\lambda}(T) + kT \ln \rho + kT a \rho (1 - a\rho)^{-1} - kT \ln (1 - a\rho) \quad (33)$$

Hence,

$$\ln \gamma = \lambda^{\text{ex}}/kT = a\rho (1 - a\rho)^{-1} - \ln(1 - a\rho) \quad (34)$$

Note that $\ln \gamma \rightarrow 0$ as $\rho \rightarrow 0$, as required by eq. (4).

The square-well potential is defined by the function,

$$u(x) = \infty, \quad x < a \quad (35a)$$

$$u(x) = -\epsilon, \quad a < x < a + \sigma \quad (35b)$$

$$u(x) = 0 \quad x > a + \sigma \quad (35c)$$

where ϵ is positive. These molecules are mutually attractive hard spheres of diameter a . The square-well potential qualitatively simulates the combination of short-range repulsive core and longer-range attractive force characteristic of all real, electrically neutral molecules. To satisfy condition (4) for the validity of Gürsey's treatment, the range σ of the attractive force should be chosen not greater than a . By substituting eq. (35) in eq. (20), Gürsey found that

$$\phi(T, \pi) = (kT/\pi) e^{-\pi a/kT} e^{\epsilon/kT} \times [1 - e^{-\pi \sigma/kT} (1 - e^{-\epsilon/kT})] \quad (36)$$

Then, from eq. (20), he derived the equation of state,

$$\pi = \rho kT [1 - a\rho + \sigma\rho (1 - e^{-\epsilon/kT}) \times (e^{\pi \sigma/kT} - 1 + e^{-\epsilon/kT}) - 1]^{-1} \quad (37)$$

The structure of this equation becomes more apparent if one notes that it reduces to eq. (32) for a pure hard-sphere fluid if either the attractive range σ or the well depth ϵ is set equal to zero. It is convenient to define the reduced variables

$$\rho' = a\rho, \quad \pi' = \pi a/kT, \quad \sigma' = \sigma/a, \quad \epsilon' = \epsilon/kT, \quad (38)$$

and

$$\alpha = 1 - e^{-\epsilon'}. \quad (39)$$

Then, from eqs. (24) and (36),

$$\ln \gamma = \pi' - \epsilon' - \ln[(\rho'/\pi')(1 - \alpha e^{-\pi'\sigma'})], \quad (40)$$

where π' must be calculated from eq. (37) in its reduced form,

$$\pi' = \rho' [1 - \rho' + \sigma' \rho' \alpha (e^{\pi'\sigma'} - \alpha)^{-1}]^{-1}. \quad (41)$$

Note that eq. (41) is transcendental in π' , so that an explicit analytical solution cannot be given.

2.3. Numerical analysis of δ

Recall that δ is defined by eq. (17). For the pure hard-sphere potential, the exact relation

$$\delta = \rho'(2 - \rho')(1 - \rho')^{-2}, \quad (42)$$

where $\rho' = a\rho$, is derived from eq. (34). For the square-well potential, δ must be computed numerically, using a straightforward iterative procedure to evaluate π' from eq. (41), with $\pi' = 0$ as the first trial value. The results are given in table 1, where $\delta(i, j)$ means that δ is evaluated for the fixed parameters $\sigma' = i$ and $\epsilon' = j$. For the case of non-attractive hard spheres, $\delta(0, 0)$, the diffusion constant D at $a\rho = 0.3$ is twice as large as the tracer diffusion, or self-diffusion, constant D^* at the same concentration (eq. (16)); while at $a\rho = 0.7$ it is an order of magnitude larger, and at $a\rho = 0.9$, two orders of magnitude larger. For the square-well poten-

tial, D is smaller than D^* at low density; if $a\rho = 0.1$, D is 67% of D^* for $(\sigma', \epsilon') = (1, 2)$, which simulates a van der Waals attraction ($\epsilon \approx 1$ kcal/mole at 25°C), while for $\epsilon' = 5$ ($\epsilon \approx 2.5$ kcal/mole, still a weak force), D is only 16% of the value of D^* .

The maximum value of ρ' is unity, which corresponds to closest packing of the hard spheres. According to eq. (42), then, the maximum value of δ is ∞ , attained when $\rho' = 1$. If $\epsilon' \rightarrow \infty$ with ρ' fixed, it may be verified analytically that $\delta \rightarrow -1$ from eqs. (40) and (41), since $\ln \gamma$ becomes proportional to $-\ln \rho'$. The general inequality (18) is therefore obeyed.

2.4. Tracer diffusion in a concentration gradient

In the previous section the effect of molecular interactions on single-file transport was seen to be reflected in exceedingly large departures of the diffusion coefficient D , which governs the flux driven by a macroscopic gradient $d\rho/dx$, from the tracer diffusion coefficient D^* , which characterizes purely random thermal motion when $d\rho/dx = 0$. An equivalent, but instructive, description may be obtained by the comparison of the *effective* tracer diffusion coefficient in the presence of a macroscopic gradient with the actual tracer diffusion coefficient (when $d\rho/dx = 0$). To fix definitions, let us agree to set

$$j^* = -D_{\text{eff}}^*(d\rho^*/dx), \quad (43)$$

where j^* is the flux of tracer species and $d\rho^*/dx$ its concentration gradient, regardless of whether or not $d\rho/dx$, the macroscopic gradient, equals zero. If $d\rho/dx = 0$, D_{eff}^* is, by definition, equal to D^* ; if molecular interactions are negligible, either because the system is sufficiently dilute or because it is "ideal," D_{eff}^* must also equal D^* even if $d\rho/dx \neq 0$. An explicit formula for D_{eff}^* will now be derived for the general case.

The tracer flux j^* is the superposition of the contribution driven by random thermal motion with that due to any net directed force f ,

$$j^* = -\frac{kT}{\zeta} \frac{d\rho^*}{dx} + \frac{\rho^*}{\zeta} f. \quad (44)$$

In the absence of externally applied forces but in the presence of a macroscopic concentration gradient $d\rho/dx$, f will, as explained earlier, be given by $-kT d \ln \gamma/dx$. Hence,

Table 1
Deviation δ from the Einstein relation

ρ'	$\delta(0, 0)$	$\delta(1, 2)$	$\delta(1, 3)$	$\delta(1, 5)$
0.1	0.24	-0.33	-0.57	-0.84
0.3	1.04	-0.20		
0.5	3.00	0.79		
0.7	10.1	12.7		
0.9	99.0			

$$\begin{aligned}
 j^* &= -\frac{kT}{\xi} \frac{d\rho^*}{dx} - \frac{kT}{\xi} \rho^* \frac{d \ln \gamma}{d\rho} \frac{d\rho}{dx} \\
 &= -\frac{kT}{\xi} \left(1 + \frac{\rho^*}{\rho} \frac{d \ln \gamma}{d \ln \rho} \frac{d\rho/dx}{d\rho^*/dx} \right) \frac{d\rho^*}{dx} \\
 &= -\frac{kT}{\xi} \left(1 + \delta \frac{d \ln \rho}{d \ln \rho^*} \right) \frac{d\rho^*}{dx}. \quad (45)
 \end{aligned}$$

Therefore, from eqs. (13) and (43),

$$D_{\text{eff}}^*/D^* = 1 + \delta (d \ln \rho / d \ln \rho^*). \quad (46)$$

To interpret this relation, suppose first that the tracer gradient is along the macroscopic gradient and proportional to it, $\rho^*(x) = c \rho(x)$, where c is a constant orders of magnitude less than unity. Then

$$D_{\text{eff}}^*/D^* = 1 + \delta, \quad (47)$$

i.e., D_{eff}^* becomes identical to the macroscopic diffusion coefficient D . From table 1, it can be seen that D_{eff}^* must then be positive, so that tracer flows along its gradient, but that D_{eff}^* can be much greater or much less than D^* , depending upon whether the intermolecular potential is predominantly repulsive or attractive. The macroscopic gradient exerts a profound effect on the tracer flow.

If the tracer gradient is along the macroscopic gradient, but not in proportion to it, for example, if $\rho^* = c\sqrt{\rho}$, then

$$D_{\text{eff}}^*/D^* = 1 + 2\delta, \quad (48)$$

and, for an intermolecular potential with a sufficiently strong attractive component, table 1 indicates that D_{eff}^* can be negative, the tracer flow then being in opposition both to its own gradient and the macroscopic gradient.

Finally, if the tracer gradient is opposed to the macroscopic gradient, say, $\rho^* = c/\rho$, then

$$D_{\text{eff}}^*/D^* = 1 - \delta, \quad (49)$$

and, according to table 1, D_{eff}^* can have a very large negative value if the intermolecular potential is primarily of repulsive nature; the tracer species is swept along by the macroscopic flow in opposition to the tracer gradient.

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